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On The Unique Solvability of The Inverse Problem Of Magneto-Electro-Encephalography

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Abstract:

Contrary to the prevailing opinion about the incorrectness of the inverse MEEG-problem, its unique solvability is established within the framework of the system of Maxwell's equations [1]. The solution of this problem is the distribution of $\mathbf{y} \mapsto \mathbf{q}(\mathbf{y})$ *current dipoles of neurons of the brain, occupying the region* $Y \subset \mathbb{R}^3$*. The initial data are non-invasive measurements of electric and magnetic fields induced by current dipoles. The solution has the form* $\mathbf{q} = \mathbf{q}_0 + \mathbf{p}_0\,\delta|_{\partial Y}$*, where* $\mathbf{q}_0$ *is the usual function defined in Y, and* $\mathbf{p}_0\,\delta|_{\partial Y}$ *is a δ-function on the boundary of the domain Y with a certain density* $\mathbf{p}_0$*. This result was obtained due to the fact that: 1) Maxwell's equations are taken as a basis; 2) a transition was made to the equations for the potentials of the magnetic and electric fields; 3) the theory of boundary value problems for elliptic pseudo-differential operators with an entire index of factorization is used. This allowed us to find the correct functional class of solutions of the corresponding integral equation of the first kind. Namely: the solution has a singular boundary layer in the form of a delta function (with some density) at the boundary of the domain. From the point of view of the MEEG problem, this means that the sought-for current dipoles are also concentrated in the cerebral cortex.*

Keywords: inverse problem, Maxwell's equations, integral equation of the first kind

1. The inverse MEEG problem is the problem of finding the distribution of the current dipoles $\mathbf{q}: Y \to \mathbb{R}^3$ (current dipole moments) in neurons of the brain, occupying the set $Y \subset \mathbb{R}^3$. In this case, the initial information is given by the magnetic data B=μH, as well as the electric D=εE induction measured on the surface X, which is the inside of the helmet worn on the head with SQUID sensors (Superconducting quantum Interference device) [2, 3].

The parameters μ>0 and ε=ε(x)>0 are magnetic and dielectric permeabilities. For bio-means $\mu \approx \mu_0$ (see http://ilab.xmedtest.net/?Q=node/3521), where μ_0 is the magnetic permeability of the vacuum. However, we will not assume that μ=const, so as not to exclude other applications, including problems of magnetic microscopes [4], etc.

We shall start from the Maxwell equations

$$\begin{aligned} &\mu\partial_t H(\mathbf{x},t) + \text{rot } E(\mathbf{x},t) = 0, \qquad \text{div } B(\mathbf{x},t) = 0, \\ -&\varepsilon(\mathbf{x})\partial_t E(\mathbf{x},t) + \text{rot } H(\mathbf{x},t) = \mathbf{J}^v(\mathbf{x}) + \mathbf{J}^p(\mathbf{x}), \qquad \text{div} D(\mathbf{x},\mathbf{t}) = \rho. \end{aligned} \tag{1}$$

Here $\mathbf{J}^v = \sigma E$ is the so-called volumetric or, as they say, ohmic current (more precisely, its density), because it is subordinate to Ohm's law associated with the coefficient of electrical conductivity $\sigma = \sigma(\mathbf{x}) \geq 0$, which is assumed to be independent of t. The volume current is the result of the action of a macroscopic electric field on the charge carriers in the conducting medium of the brain. Neuronal same activity causes the so-called primary (they say also, principal) current $\mathbf{J}^p(\mathbf{x})$. It arises as a result of dielectric polarization and it represents a movement of charges inside or near the cell. The volume density of these charges is denoted by ρ. Particles possessing these charges are part of the molecules. They are displaced from their equilibrium positions under the action of an external electric field, without leaving the molecule into which they enter.

Essential is the circumstance, especially noted in the fundamental work [2]. It is related to the frequency ratio ω of the oscillations of the electromagnetic field $H(\mathbf{x},t) = \mathbf{H}(\mathbf{x})e^{i\omega t}$, $E(\mathbf{x},t) = \mathbf{E}(\mathbf{x})e^{i\omega t}$, and the frequency of electrical oscillations in brain cells. The analysis in [2] (on page 426) shows that the quasi-static approximation for the (1) system is justified. There, on the same page, is additionally noted: "A current dipole **q**, approximating a localized primary current, is a widely used concept in neuro-magnetism…In EEG and MEG applications, a current dipole is used as an equivalent source for the unidirectional primary current that may extend over several square centimetres of cortex."

As a result, we arrive at the following equations

$$\text{rot } \mathbf{E} = 0, \qquad \text{rot } \mathbf{H} = \sigma\mathbf{E} + \mathbf{q}, \qquad \text{div } \mathbf{B} = 0, \qquad \text{div } \mathbf{D} = \rho. \tag{2}$$

2. As is known,

$$\text{rot } \mathbf{E} = 0 \quad \Leftrightarrow \quad \mathbf{E} = -\nabla\Phi \qquad \text{and} \qquad \text{div } \mathbf{B} = 0 \quad \Leftrightarrow \quad \mathbf{B} = \text{rot } \mathbf{A} \tag{3}$$

Since $\text{div}\left(\varepsilon\mathbf{E}\right) = \rho$, then

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$$-\varepsilon\Delta\Phi - \nabla\varepsilon\nabla\Phi = \rho. \tag{4}$$

According to physical ideas, the potential $\Phi \overset{(4)}{=} \Phi_\rho$ of the field $\mathbf{E}_\rho = -\nabla\Phi$ at infinity is a constant that can be considered equal to zero. For similar reasons, the vector potential $\mathbf{A}$ of the field $\mu\mathbf{H} = \mathrm{rot}\mathbf{A}$ is also chosen to be zero at infinity. Since

$$\mathrm{rot}\left(\mathrm{rot}\,\mathbf{A}\right) = \nabla\mathrm{div}\,\mathbf{A} - \Delta\mathbf{A}, \qquad \mathrm{rot}(\mu\mathbf{H}) = \mu\mathrm{rot}\mathbf{H} - \mathbf{H}\times\nabla\mu \tag{5}$$

then $\Delta\mathbf{A} = -\mathrm{rot}\,(\mu\mathbf{H}) + \nabla\mathrm{div}\,\mathbf{A} = -\mu\mathrm{rot}\mathbf{H} + \mathbf{H}\times\nabla\mu + \nabla\mathrm{div}\mathbf{A}$. But $\mathrm{rot}\,\mathbf{H} = \sigma\mathbf{E} + \mathbf{q}$, and $\mathbf{E} = -\nabla\Phi$. Therefore, putting $\mathbf{Q} = \mu\mathbf{q}$, $\sigma_\mu = \sigma\mu$, we have

$$\Delta\mathbf{A} = -\mathbf{Q} + \nabla[\sigma_\mu\Phi + \mathrm{div}\mathbf{A}] - \Phi\nabla\sigma_\mu + \mathbf{H}\times\nabla\mu. \tag{6}$$

The vector potential $\mathbf{A}$ is determined up to a potential field. Indeed, we have: $\mathrm{rot}\left(\mathbf{A} - \mathbf{A}^*\right) = 0 \overset{(3)}{\Leftrightarrow} \mathbf{A} - \mathbf{A}^* = \nabla\varphi$, i.e. $\mathbf{A} = \mathbf{A}^* + \nabla\varphi$, where φ is some function. Taking φ as solution of the equation $\Delta\varphi = -\mathrm{div}\,\mathbf{A}^* - \sigma_\mu\Phi$, subjected to the condition $\varphi\big|_\infty = 0$ (since $\mathbf{A}^*\big|_\infty = 0$, $\Phi\big|_\infty = 0$), we obtain

$$\Delta\mathbf{A}\left(\mathbf{x}\right) = -\mathbf{F}\left(\mathbf{x}\right), \qquad \text{where} \qquad \mathbf{F}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) + \Phi_\rho(\mathbf{x})\nabla\sigma_\mu(\mathbf{x}) - \mathbf{H}(\mathbf{x})\times\nabla\mu(\mathbf{x}). \tag{7}$$

Assuming $\mathbf{a} = \left(a_1, a_2, a_3\right)$, where $\Delta a_j\left(\mathbf{x}\right) = \delta\left(\mathbf{x}\right)$, $a_j\left(\infty\right) = 0$, i.e. $a_j\left(\mathbf{x}\right) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}$, we get

$$\Delta\mathbf{A}\left(\mathbf{x}\right) \overset{(7)}{=} -\int\mathbf{F}\left(\mathbf{y}\right)\Delta a\left(\mathbf{x}-\mathbf{y}\right)d\mathbf{y} = \Delta\left[-\int\mathbf{F}\left(\mathbf{y}\right)a\left(\mathbf{x}-\mathbf{y}\right)d\mathbf{y}\right].$$

From here

$$\mathbf{A}\left(\mathbf{x}\right) = \frac{1}{4\pi}\int\mathbf{F}\left(\mathbf{y}\right)\frac{1}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y} \overset{(7)}{=} \frac{1}{4\pi}\int\left(\mathbf{Q}\left(\mathbf{y}\right) + \Phi_\rho\left(\mathbf{y}\right)\nabla\sigma\left(\mathbf{y}\right) - \mathbf{H}(\mathbf{y})\times\nabla\mu(\mathbf{y})\right)\frac{1}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}, \tag{8}$$

since the Laplace equation has a unique solution that vanishes at infinity (as already noted, $\mathbf{A}\big|_\infty = 0$).

3. As a result, we obtain an integral equation of the I-kind

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$$\int_Y \frac{\mathbf{Q}(\mathbf{y})\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in Y, \tag{9}$$

whose right-hand side, given by the formula

$$\mathbf{f}(\mathbf{x}) = 4\pi \mathbf{A}(\mathbf{x}) - \int_Y \frac{\Phi(\mathbf{y})\nabla\sigma(\mathbf{y})\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \tag{10}$$

is completely determined by the fields $\mathbf{E} = -\nabla\Phi_\rho$ and $\mathbf{B} = \text{rot}\,\mathbf{A}$.

We can assume [2, 3], that the components of the vector $\mathbf{f}$ are sufficiently smooth, in any case, belongs to the Sobolev space $W_2^s(Y)$, where $s > 3/2$.

Theorem 1 (see [5]). The equation (9) is uniquely solvable, and its solution has the form

$$\mathbf{Q}(\mathbf{x}) = \mathbf{Q}_0(\mathbf{x}) + \mathbf{P}_0(\mathbf{y}')\delta|_{\partial Y} \in W_2^{s-2}(Y) + W_2^{s-1}(\partial Y) \otimes \delta|_{\partial Y}, \tag{11}$$

where $\delta|_{\partial Y}$ is the δ–function on ∂Y.

If there is measurement data of the fields $\mathbf{E} = -\nabla\Phi_\rho$ and $\mathbf{B} = \text{rot}\,\mathbf{A}$ to a finite collection of points $\mathbf{x}_k$ we can recover them to some extent, solving the minimization problem of the functional of the following type

$$G(\rho, \mathbf{A}) \overset{def}{=} \left\| \mathbf{E}(\mathbf{x}_k) + \nabla\Phi_\rho|_{\mathbf{x}=\mathbf{x}_k} \right\|^2 + \left\| \mathbf{B}(\mathbf{x}_k) - \text{rot}\,\mathbf{A}\,|_{\mathbf{x}=\mathbf{x}_k} \right\|^2 \tag{12}$$

provided that $\text{div}\,\mathbf{B} = 0$, $\text{rot}\,\mathbf{E} = 0$.

Of course, in any case, these fields will be restored with some error. Its minimization can be achieved by the use of methods of optimal interpolation (see, for example, [6-8]).

We also note that in [9] a connection is established between the solution $\mathbf{Q}$ of the integral equation (9) and the solution $\mathbf{u}$ of an integral equation of the second kind

$$\eta^2 \mathbf{u}(\mathbf{x}) + \int_Y \frac{\mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y} = \mathbf{f}(\mathbf{x}), \qquad \eta > 0. \tag{13}$$

Theorem 2 (see [9]). *The solution of equation (9) is representable in the form*

$$\mathbf{u}(\mathbf{x}) = \mathbf{Q}_0(\mathbf{x}) + \frac{1}{\eta}\mathbf{P}_0(\mathbf{y}')\varphi e^{-y_n/\eta} + \mathbf{r}_0(\mathbf{x}, \eta), \tag{14}$$

where $\|r_0\|_{L^2} \le C\sqrt{\eta}$, y_n *is the distance along the normal from* **x** to $\mathbf{y} \in \Gamma$, and $\varphi \in C^\infty(\bar{Y})$, $\varphi \equiv 1$ in a small neighbourhood ∂Y and $\varphi \equiv 0$ outside a larger neighbourhood.

The formula (14) can serve as a basis for the numerical solution of the problem (9).

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