Characterizations of Quaternionic Bertrand Curves of A Helix In semi-Euclidean Space

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Abstract — In this study, we define a quaternionic Bertrand curves of a helix in semi-Euclidean space $E_2^4$ and we investigate its properties for two cases. In the first case; we obtain the necessary and sufficient conditions for the Bertrand partner curves. In the other case, we obtain Bertrand curves of a helix for a quaternionic curve in semi-Euclidean space $E_2^3$ and denoted the Frenet apparatus by $\{T; N; B; E_1\}$. We say that $\alpha$ is a helix (general helix), if its tangent vector $T$ makes a constant angle with a fixed direction $U$. In addition, we gave some theorems for a quaternionic curves to be a quaternionic Bertrand curve of a helix.

Keywords — Semi-Euclidean spaces; Quaternion algebra; Quaternionic frame; Quaternionic Bertrand curves.

I. INTRODUCTION

The quaternions first described by Sir William R. Hamilton in 1843 as a number system that extends the complex numbers. Whereas a standard complex number has a scalar component and an imaginary component, with quaternions the imaginary part is an imaginary vector based on three imaginary orthogonal axes. The quaternions are both relatively simple and very effective for rotations. So, the quaternion algebra has played a significant role recently in several areas of the physical science; namely, in differential geometry, in analysis and synthesis of mechanism and machines, simulation of particle motion in molecular physics and quaternionic formulation of equation of motion in theory of relativity, ([1],[2]). In 1888, C. Bioche give a new theorem in [3] to obtaining Bertrand curves by using the given two curves $C_1$ and $C_2$ in Euclidean 3-space. Later, in 1960, J. F. Burke in [4] give a theorem related with Bioche's theorem on Bertrand curves. In 1987, The Serret-Frenet formulae for a quaternionic curves in $R^3$ are introduced by K. Bharathi and M. Nagaraj. Moreover, they obtained the Serret-Frenet formulae for the quaternionic curves in $R^4$ by the formulae in [5]. Then, lots of studies have been published by using this studies. One of them is A. C. Coken and A. Tuna's study ([6],[7]) which they gave Serret-Frenet formulas, inclined curves, harmonic curvatures and some characterizations for a quaternionic curve in the semi-Euclidean spaces $E_2^4$; A helix in semi-Euclidean space is a curve whose tangent line makes a constant angle with a fixed direction. Helices are well known curves in classical differential geometry of space curves ([11],[12]) and we refer to the reader for recent studies on this type of curves ([11],[12]). Recently, Slant helices have been defined by Izumiya and Takeuchi and they gave a characterization of slant helices in Euclidean 3-space $E_3^3$ with a proposition [16]. After that, L. Kula and Y. Yayli studied spherical images, the tangent indicatrix and the binormal indicatrix of a slant helix and they obtained that the spherical images are spherical helices [17]. Furthermore, many geometers have studied helices in Euclidean space and in Minkowski space. A Magden obtained an integral characterization of helices in Euclidean 4-space $E_3^4$ [10] and corresponding characterization of helices in Minkowski 4-space $E_3^4$ was given by H. Kocayigit and M. Onder [13]. Gok and Kahraman defined a new kind of slant helix, which they called 3-type slant helix and they gave some characterizations of this slant helix in $E_2^4$. 
A unit speed semi-real quaternionic curve is called a 3-type slant helix if its second binormal unit vector makes a constant angle with a fixed direction in a unit vector. Let and be quaternionic curves in with arc-length parameters and , respectively. Then is called a Bertrand curve, and a Bertrand partner curve of if principal normal vector fields of the curves and are linearly dependent and also is called a quaternionic Bertrand curve if and only if

\[ \lambda \varepsilon_N \varepsilon_t \kappa(s) + \mu \varepsilon_N \tau(s) = 1, \]

where \( \lambda \) and \( \mu \) are real constants and \( \kappa(s) \) is the principal curvature, \( \tau(s) \) is the torsion of the curve \( \alpha \).

In this work, we define characterizations of semi-real quaternionic Bertrand curves in the four-dimensional semi-Euclidean space with vanishing curvatures form. In order to do this, we study the Serret-Frenet formulae of the curve in and then applying quaternionic Bertrand curves. We obtain the families of quaternionic Bertrand curves in . And then we studied a new kind of Bertrand curves of a helix in semi-Euclidean space and we gave Frenet formulae for the quaternionic curve in semi-Euclidean space. Then we defined a general helix and slant helix. Also, we obtained some theorems for a quaternionic Bertrand curves of a helix.

II. PRELIMINARIE

Let \( Q_v \) be the four-dimensional vector space over a field \( v \) whose characteristic greater than 2. Let \( e_i \) ( \( 1 \leq i \leq 4 \)) be a basis for the vector space. Let the rule of multiplication on \( Q_v \) be defined on \( e_i \) and extended to the whole of the vector space distributivity as follows ([6], [7]):

A semi-real quaternion is defined by \( q = ae_1 + be_2 + ce_3 + d \) (or \( S_q = d \) and \( V_q = ae_1 + be_2 + ce_3 \)). Then a quaternion \( q \) can now write as \( q = S_q + V_q \), where \( S_q \) and \( V_q \) are the scalar part and vectorial part of \( q \), respectively. Such that \( e_i \times e_j = -e_k e_i e_j \), where \((ijk)\) is an even permutation of \((123)\) in semi-Euclidean space \( E_4 \). Notice here that we define the set of all semi-real quaternions by \( Q_v \):

\[ Q_v = \{ q \mid q = ae_1 + be_2 + ce_3 + d ; a, b, c, d \in R \ and \ e_1, e_2, e_3 \in R^3 \}. \]

We put, Using these basic products we can now expand the product of two quaternions to give

\[ p \times q = S_p S_q + ( V_p, V_q ) + S_p V_q + S_q V_p + V_p V_q \]

for every \( p, q \in Q_v \):

We see that the quaternionic product contains all the products of semi-Euclidean space \( E_4 \). There is a unique involutory anti-automorphism of the quaternion algebra, denoted by the symbol \( \gamma \) and defined as follows:

\[ \gamma q = -ae_1 - be_2 - ce_3 \]

for every \( q = ae_1 + be_2 + ce_3 + d \in Q_v \).

which is called the Hamiltonian conjugation. This defines the symmetric non-degenerate valued bilinear form \( h \) as follows:

\[ h(p, q) = \frac{1}{2} [(p \times \gamma q) + (q \times \gamma p)] \]

for every \( p, q \in Q_v \), the norm of semi-real quaternion \( q \) is denoted by

\[ \| q \|^2 = h_v(q, q) = |(q \times \gamma q)| = |a^2 + a^2 + c^2 + d^2| \]
for \( p, q \in Q_v \) where if \( h_v(p, q) = 0 \) then \( p \) and \( q \) are called \( h \)-orthogonal. The concept of a spatial quaternion will be made use throughout our work. \( q \) is called a spatial quaternion whenever \( q + \gamma q = 0 ([6]-[8]) \).

**Definition 1:** The 4-dimensional semi-Euclidean spaces in \( E_2^4 \) are identified with the spaces of unit quaternions. Let

\[
\alpha : I \subset \mathbb{R} \rightarrow Q_v
\]

be a smooth curve in \( E_2^4 \). Let the parameter \( s \) be chosen such that the tangent \( T(s) = \alpha'(s) \) has unit magnitude. Let \( \{T; N; B; B_1\} \) be the Frenet apparatus of the differentiable Euclidean space curve in the Euclidean spaces \( E^4 \). Then Frenet formulas are given by

\[
\begin{align*}
T'(s) &= e_N \kappa(s) N(s) \\
N'(s) &= e_N \tau(s) B(s) - e_N e_i \kappa(s) T(s) \\
B'(s) &= -e_i \tau(s) N(s) + e_i [\sigma - \kappa e_N e_i e_T] (s) B_1(s) \\
B_1'(s) &= -e_i [\sigma - \kappa e_N e_i e_T] (s) B(s)
\end{align*}
\]

where \( \kappa(s) = e_N ||T'(s)|| \) and \( N'(s) = ||e_N|| \).

**III. QUATERNIONIC BERTRAND CURVES IN SEMI-EUCLIDEAN SPACE**

**Definition 2.** Let \( E_2^4 \) be the four-dimensional semi-Euclidean space with inner product \( h_v(p, q) \) for every \( p, q \in Q_v \). If there exists a corresponding relationship between the quaternionic curves \( \alpha \) and \( \alpha^* \) such that, at the corresponding points of the quaternionic curves, the principal normal vector of \( \alpha \) coincides with normal vector of \( \alpha^* \), then \( \alpha \) is called a Bertrand curve, and \( \alpha^* \) a Bertrand partner curve of \( \alpha \). The pair \( \{\alpha, \alpha^*\} \) is said to be a Bertrand pair.

**Definition 3.** Let \( \alpha(s) \) and \( \alpha^*(s^*) \) be quaternionic curves in \( E_2^4 \) with arc-length parameter \( s \) and \( s^* \), respectively. \( \{T(s), N(s), B(s), B_1(s)\} \) and \( \{T^*(s^*), N^*(s^*), B^*(s^*), B_1^*(s^*)\} \) are Frenet frames of \( \alpha \) and \( \alpha^* \), respectively. If the pair \( \{\alpha, \alpha^*\} \) are a Bertrand pair, \( N(s) \) and \( N^*(s^*) \) are linearly dependent. We can write

\[
\alpha^*(s) = \alpha(s) + \lambda(s) N(s).
\]

**Definition 4.** A unit speed semi-real quaternionic curve \( \alpha : I \subset \mathbb{R} \rightarrow E_2^4 \) non-zero curvatures \( \kappa(s), \tau(s) \) and \( [\tau - \kappa e_N e_T e_I] \) is called general or cylindrical helix if its unit tangent vector \( T \) makes a constant angle \( \phi \) with a fixed direction in a unit vector \( U \); that is \( h(T; U) = \cos \phi \) is constant along the curve \([10]; [14]\).

**Definition 5.** A unit speed semi-real quaternionic curve \( \alpha \) is called slant helix if its unit principal normal vector \( N \) makes a constant angle \( \phi \) with a fixed direction in a unit vector \( U \); that is \( h(N; U) = \cos \phi \) is constant along the curve \([16]\).

**Definition 6.** Let \( a = (a_1; a_2; a_3; a_4); b = (b_1; b_2; b_3; b_4) \) and \( c = (c_1; c_2; c_3; c_4) \) be vectors in \( E_2^4 \). The vector product in \( E_2^4 \) defined by the determinant
where $e_1$, $e_2$, $e_3$ and $e_4$ are mutually orthogonal vectors (coordinate direction vectors) satisfying equations $e_1 \wedge e_2 \wedge e_3 = e_4$, $e_2 \wedge e_3 \wedge e_4 = e_1$, $e_3 \wedge e_4 \wedge e_1 = -e_2$, $e_4 \wedge e_1 \wedge e_2 = -e_3$.

**Theorem 1.** Let $\alpha(s)$ be a quaternionic curves in semi-Euclidean space $\mathbb{E}_2^4$ with arc-length parameter $s$. Then $\alpha^*(s)$ is the Bertrand partner curve of $\alpha(s)$. Then corresponding points are a fixed distance for all $s \in I$.

**Proof:** Suppose that $\alpha(s) : I \subset \mathbb{R} \rightarrow \mathbb{E}_2^4$ is a quaternionic curve. Taking the derivate of (2) with respect to $s$ and apply the Frenet formulas, we obtain

$$\frac{d\alpha^*(s^*)}{ds^*} = \frac{ds^*}{ds} [1 - \varepsilon_N \varepsilon_t \lambda(\alpha(s)) \kappa(\alpha(s))] T(s) + \lambda'(s) N(s) + \varepsilon_n \tau(s) B(s)$$

$$T^*(s^*) = \frac{ds}{ds^*} \left[ [1 - \varepsilon_N \varepsilon_t \lambda(\alpha(s)) \kappa(\alpha(s))] T(s) + \lambda'(s) N(s) + \varepsilon_n \tau(s) B(s) \right]$$

And

$$h(T^*(s^*), N^*(s^*)) = \left\{ \frac{ds}{ds^*} \left[ [1 - \varepsilon_N \varepsilon_t \lambda(\alpha(s)) \kappa(\alpha(s))] h(T(s), N^*(s^*)) + \lambda'(s) h(N(s), N^*(s^*)) + \varepsilon_n \tau(s) h(B(s), N^*(s^*)) \right] \right\}$$

Since $N(s)$ is coincident with $N^*(s^*)$ in direction " a linearly dependent set ", we get $\lambda'(s) = 0$. This means that $\lambda(s)$ is a none zero constant function on $I$. Then

$$d(\alpha^*(s), \alpha(s)) = ||\alpha^*(s) - \alpha(s)|| = ||\lambda(s) N(s)|| = \lambda.$$  

Where $\lambda = \text{constant}$. This completes the proof.

**Theorem 2.** Let $\alpha(s)$ be a quaternionic curves in $\mathbb{E}_2^4$ with arc-length parameter $s$ and $\alpha^*(s^*)$ is a Bertrand partner curve of $\alpha(s)$. Then the angle between the tangent vector of quaternionic curve $\alpha(s)$ and $\alpha^*(s)$ is constant.

**Proof:** Let $\alpha(s)$ and $\alpha^*(s^*)$ be quaternionic curves in $\mathbb{E}_2^4$ with arc-length $s$ and $s^*$, respectively. Let's consider that

$$h(T(s), T^*(s^*)) = \cos \theta.$$  

Taking the derivate of (2) with respect to $s$, we obtain

$$\frac{d}{ds} h(T(s), T^*(s^*)) = h \left( \frac{dT(s)}{ds}, T^*(s^*) \right) + h \left( T(s), \frac{dT^*(s^*)}{ds} \right)$$

$$= h(\varepsilon_N \varepsilon_t \kappa(s) N(s), T^*(s^*)) + h \left( T(s), \varepsilon_N \varepsilon_t \kappa(s^*) N^*(s^*) \frac{ds^*}{ds} \right) = 0.$$
This means that \( h(T(s), T'(s')) = \text{constant} \). This completes the proof.

**Theorem 3.** Let \( \alpha(s) \) and \( \alpha^*(s) \) be quaternionic curves in \( \mathbb{E}_2^4 \) with arc-length parameter \( s \) and \( s^* \), respectively. Then \( \alpha \) is a quaternionic Bertrand curve if and only if

\[
\lambda \varepsilon_n \varepsilon_k \kappa(s) + \mu \varepsilon_n \tau(s) = 1,
\]

where \( \lambda \) and \( \mu \) are real constants and \( \kappa(s) \) is the principal curvature, \( \tau(s) \) is the torsion of the curve \( \alpha \).

**Proof.** Let \( \alpha^*(s^*) \) be a quaternionic Bertrand partner curve of \( \alpha(s) \). Then we can write

\[
\alpha^*(s) = \alpha(s) + \lambda(s)N(s).
\]

Taking the derivative of the last equality considering \( \psi: I \rightarrow I^* \), \( \psi(s) = s^* \) is a \( C^\infty \) function we have

\[
T^*(\psi(s)) = \frac{ds}{ds^*} 
\left[ \left[ 1 - \varepsilon_n \varepsilon_k \lambda(s) \kappa(s) \right] T(s) + \varepsilon_n \lambda(s) \tau(s) B(s) \right].
\]

If we consider the following equation

\[
T^*(\psi(s)) = \cos \theta T(s) + \sin \theta B(s)
\]

we get

\[
\cos \theta = \left( 1 - \varepsilon_n \varepsilon_k \lambda(s) \kappa(s) \right) \frac{ds}{ds^*},
\]

\[
\sin \theta = \varepsilon_n \tau(s) \lambda(s) \frac{ds}{ds^*}.
\]

Then by taking \( \frac{\cos \theta}{\sin \theta} = \mu \), we have

\[
\lambda \varepsilon_n \varepsilon_k \kappa(s) + \mu \varepsilon_n \tau(s) = 1.
\]

On the other hand, Suppose that \( \alpha(s) \) be quaternionic curves in \( \mathbb{E}_2^4 \) with curvature function \( \kappa \), \( \tau \neq 0 \) satisfying the relation \{ \( \lambda \varepsilon_n \varepsilon_k \kappa(s) + \mu \varepsilon_n \tau(s) = 1 \) \} for constant \( \lambda, \mu \). Then we can write

\[
\alpha^*(s) = \alpha(s) + \lambda(s)N(s).
\]

By taking the derivative of the last equality with respect to \( s \) and applying the Frenet formulas, we have

\[
\frac{d\alpha^*(s)}{ds^*} = \frac{ds}{ds^*} 
\left[ \left[ 1 - \varepsilon_n \varepsilon_k \lambda(s) \kappa(s) \right] T(s) + \varepsilon_n \lambda(s) \tau(s) B(s) \right]
\]

\[
= \frac{ds}{ds^*} 
\left[ \left[ 1 - \varepsilon_n \varepsilon_k \lambda(s) \kappa(s) \right] T(s) + \varepsilon_n \lambda(s) \tau(s) B(s) \right].
\]

From the hypothesis, we have \( \mu \varepsilon_n \tau(s) = 1 - \lambda \varepsilon_n \varepsilon_k \kappa(s) \) thus we get

\[
\frac{d\alpha^*(s)}{ds^*} = T'(s^*) = \frac{ds}{ds^*} \left[ \varepsilon_n \tau(s) \left[ \mu T(s) + \lambda B(s) \right] \right]
\]

Where,
Differentiating of the last equality with respect to $s$ and applying the Frenet formulas, we have

$$\frac{d\tau}{ds^*} = \frac{\cos\theta}{\mu \varepsilon_n \tau(s)}$$

and

$$\frac{d^{2}\tau}{ds^*} = \frac{\cos^2\theta}{\mu^2 \varepsilon_n \tau(s)^2}.$$
where multiplying both sides of the last equality with $\cos\theta = \left[1 - \varepsilon_N \varepsilon_\tau \lambda(s') \kappa'(s') \right] \frac{ds'}{ds}$ and $\sin\theta = -\varepsilon_N \tau'(s') \frac{ds'}{ds}$, we obtain

$$\tau \tau^* = -\frac{\sin^2\theta}{(\varepsilon_N \lambda)^2} = \text{constant}. $$

$$\varepsilon_N \mu (\tau + \tau^*) + \varepsilon_N \varepsilon_\tau \lambda (\kappa + \kappa^*) = 0$$

This completes the proof.

IV. BERTRAND CURVES OF HELIX IN SEMI EUCLIDEAN SPACE $\mathbb{E}^4_2$

Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^4_2$ be a unit speed quaternionic curve defined over the interval $I = [0, 1]$ and the arc-length parameter $s$ be chosen such that the tangent $T = \alpha'(s)$ has unit magnitude, [6]. The Serret-Frenet apparatus of $\alpha$ can be written as follows:

$$T(s) = \alpha'(s)$$

$$N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$$

$$B(s) = \varepsilon_N \varepsilon_\tau \left[ B_1(s) \wedge T(s) \wedge N(s) \right]$$

$$B_1(s) = \varepsilon_N \varepsilon_\beta \varepsilon_\tau \varepsilon_\kappa \frac{\tau(s) \wedge N(s) \wedge \alpha''(s)}{\|\tau(s) \wedge N(s) \wedge \alpha''(s)\|}, \ (\eta = \pm 1)$$

and

$$\kappa(s) = \varepsilon_N \|\alpha''(s)\|, \ \tau(s) = \varepsilon_N \frac{\tau(s) \wedge N(s) \wedge \alpha''(s)}{\|\alpha''(s)\|}, \ \left(\sigma - \varepsilon_\eta \varepsilon_\tau \varepsilon_\kappa \nu\right) = \varepsilon_\beta \varepsilon_\tau \varepsilon_\kappa \frac{\langle \alpha''(s), B_1(s) \rangle}{\|\tau(s) \wedge N(s) \wedge \alpha''(s)\|}.$$ 

Theorem 5. Let $\alpha = \alpha(s)$ be a helix in $\mathbb{E}^4_2$. $\alpha^*$ is a quaternionic Bertrand partner curve of $\alpha$. $\{T(s), N(s), B(s), B_1(s)\}$ and $\{T^*(s^*), N^*(s^*), B^*(s^*), B_1^*(s^*)\}$ are Frenet frames $\alpha$ and $\alpha^*$, respectively.

Proof. We assume that $\alpha^*$ is a quaternionic Bertrand partner curve of $\alpha$. The equation (3) with respect to $s$ and using equation (2), we have

$$\left(\alpha^*\right)'(s^*) = T^*(s^*) \frac{ds^*}{ds} = \left[ (1 - \lambda \varepsilon_N \varepsilon_\tau \kappa(s)) T(s) + \lambda \varepsilon_N \tau(s) B(s) \right].$$

From the equation (6), we can obtain

$$\|\left(\alpha^*\right)'(s^*)\|^2 = \left(\frac{ds^*}{ds}\right)^2 = \left(1 - \lambda \varepsilon_N \varepsilon_\tau \kappa(s)\right)^2 + \left(\lambda \varepsilon_N \tau(s)\right)^2 \neq 0$$

and

$$\frac{ds^*}{ds} = \|\left(\alpha^*\right)'(s^*)\| = \pm \sqrt{\left(1 - \lambda \varepsilon_N \varepsilon_\tau \kappa(s)\right)^2 + \left(\lambda \varepsilon_N \tau(s)\right)^2}$$

for all $s \in I$. If we denote
it easy to obtain
\[ T^*(s^*) = \frac{1}{p} \left[ (1 - \lambda \varepsilon_N \varepsilon_t \kappa(s)) T(s) + \lambda \varepsilon_N \varepsilon_t \tau(s) B(s) \right]. \] (7)

the equation (6) with respect to s, we obtain
\[ (\alpha^*)'(s^*) = \left\{ \frac{\varepsilon_N \kappa(s) - \lambda \varepsilon_t \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_t \tau^2(s) \right)}{\varepsilon_N \kappa(s) + \lambda \varepsilon_t \varepsilon_t \tau(s) \left( \sigma - \varepsilon_t \varepsilon_N \kappa(s) \right) B_1(s)} \right\} N(s). \] (8)

Using the last equality, we have
\[ \| (\alpha^*)''(s^*) \|^2 = \frac{1}{p} \left\{ \left( \varepsilon_N \kappa(s) - \lambda \varepsilon_t \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_t \tau^2(s) \right) \right)^2 + \left[ \lambda \varepsilon_t \varepsilon_t \tau(s) \left( \sigma - \varepsilon_t \varepsilon_N \kappa(s) \right) \right]^2 \}^{1/2}. \]

If we take
\[ \begin{align*}
\alpha(s) &= \varepsilon_N \kappa(s) - \lambda \varepsilon_t \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_t \tau^2(s) \right) \\
b(s) &= \lambda \varepsilon_t \varepsilon_t \tau(s) \left( \sigma - \varepsilon_t \varepsilon_N \kappa(s) \right)
\end{align*} \]

and
\[ R(s) = \left\{ \left[ \varepsilon_N \kappa(s) - \lambda \varepsilon_t \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_t \tau^2(s) \right) \right]^2 + \left[ \lambda \varepsilon_t \varepsilon_t \tau(s) \left( \sigma - \varepsilon_t \varepsilon_N \kappa(s) \right) \right]^2 \}^{1/2}. \]

By using equation (4), we obtain the principal normal vector of the curve \( \alpha^* \)
\[ N^*(s^*) = \frac{1}{R} \left( \alpha(s) N(s) + b(s) B_1(s) \right). \] (9)

By taking the derivative of equation (8) with respect to s, we obtain
\[ (\alpha^*)'''(s^*) = \left\{ \varepsilon_t \varepsilon_N \kappa(s) \left( \lambda \left( \varepsilon_t \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_t \varepsilon_N \tau(s)^2 \right) \right) - \varepsilon_N \kappa(s) \right) \right\} T(s) \]
\[ + \left\{ \varepsilon_N \tau(s) \left( \varepsilon_N \kappa(s) - \lambda \left( \varepsilon_t \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_t \varepsilon_N \tau(s)^2 \right) \right) + \varepsilon_t \varepsilon_t \varepsilon_N \kappa(s) \right) \right\} B(s). \] (10)

for all \( s \in I \). If we denote
\[ m(s) = \varepsilon_t \varepsilon_N \kappa(s) \left( \lambda \left( \varepsilon_t \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_t \varepsilon_N \tau(s)^2 \right) \right) - \varepsilon_N \kappa(s) \), \]
\[ n(s) = \varepsilon_N \tau(s) \left( \varepsilon_N \kappa(s) - \lambda \left( \varepsilon_t \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_t \varepsilon_N \tau(s)^2 \right) \right) + \varepsilon_t \varepsilon_t \varepsilon_N \kappa(s) \right). \]

We can rewrite the equation (10) as,
\[ (\alpha^*)'''(s^*) = m(s) T(s) + n(s) B(s) \] (11)

where \( m(s) \) and \( n(s) \) are constant functions on \( I \). Now, from (7), (9) and
we can compute the vector form \( T^*(s^*) \wedge N^*(s^*) \wedge \alpha'''(s_\cdot) \) as follows:

\[
T^*(s^*) \wedge N^*(s^*) \wedge \alpha'''(s^*) = -\frac{1}{PR} \begin{vmatrix} -T & -N & B & B_1 \\ -1 - \lambda \varepsilon_N \varepsilon_T \kappa(s) & 0 & \lambda \varepsilon_N \varepsilon_T \kappa(s) & 0 \\ 0 & \alpha(s) & 0 & b(s) \\ \varepsilon_n(s) & 0 & n(s) & 0 \end{vmatrix}
\]

\[
= -\frac{S}{PR} \left[ \lambda (\varepsilon_n)^2 \tau(s)(\sigma - \varepsilon_T \varepsilon_N \kappa(s))N(s) \right] B_1(s) \right] \cdot \varepsilon_T \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_n(\tau(s))^2 \right) = \varepsilon_N \kappa(s) \left( 1 + \lambda^2 \varepsilon_T \varepsilon_N \varepsilon_b((\sigma - \varepsilon_T \varepsilon_N \kappa(s))^2) \right).
\]

From equation (11), we obtain

\[
\|T^*(s^*) \wedge N^*(s^*) \wedge \alpha'''(s^*)\| = \left( \frac{S}{PR} \right)^2 \left[ \frac{(\lambda \varepsilon_n^2 \tau(s)(\sigma - \varepsilon_T \varepsilon_N \kappa(s))^2)}{\varepsilon_N \kappa(s) - \lambda \varepsilon_T \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_n(\tau(s))^2 \right)} \right] \neq 0
\]

and then

\[
\|T^*(s^*) \wedge N^*(s^*) \wedge \alpha'''(s^*)\| = \pm \frac{S}{PR} \left\{ \frac{\varepsilon_N \kappa(s)}{\varepsilon_N \kappa(s) - \lambda \varepsilon_T \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_n(\tau(s))^2 \right)} \right\}^{1/2}.
\]

From the equation (12), we obtain

\[
B_1^*(s^*) = \frac{\varepsilon_N \kappa(s)}{\varepsilon_N \kappa(s) - \lambda \varepsilon_T \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_n(\tau(s))^2 \right)} \lambda \varepsilon_n^2 \tau(s)(\sigma - \varepsilon_T \varepsilon_N \kappa(s))N(s) B_1(s)
\]

for all \( s \in I \), if we denote

\[
K(s) = \lambda \varepsilon_n^2 \tau(s)(\sigma - \varepsilon_T \varepsilon_N \kappa(s)) \quad L(s) = \varepsilon_N \kappa(s) - \lambda \varepsilon_T \left( (\varepsilon_N \kappa(s))^2 + \varepsilon_n(\tau(s))^2 \right)
\]

we can rewrite equation (14)

\[
B_1^*(s^*) = \frac{\varepsilon_n^2 \varepsilon_T \varepsilon_N \kappa(s)}{R} [K(s)V_2(s) - L(s) B_1(s)].
\]

Now, from the equations (7), (9) and (14), we can determine the third vector field of Frenet frame
So, we obtain

\[
B^* = \eta \frac{e_n^2 e_B e_{\tau} e_N}{p} \left[ \lambda e_{n} \tau(s) T(s) + \left( 1 - \lambda e_{n} e_{\tau}(s) \right) B(s) \right].
\]

This completes the proof. We describe the following theorem by considering the equations obtained above.

**Corollary 1.** Let \( \alpha = \alpha(s) \) be a helix in \( \mathbb{E}^4 \). Moreover, \( \alpha^* \) be a quaternionic Bertrand partner curve of \( \alpha \). Then

(i) \( \alpha^* \) is also a helix.

(ii) \( \alpha^* \) can not be a generalized helix.

(iii) \( \alpha^* \) can not be a 3-type slant helix.

(iv) if \( \alpha^* \) lies on the hypersphere \( S^3 \), then, the sphere’s radius is equal to

\[
\frac{\sqrt{\tau^2 + (\sigma - e_{\tau} e_N e_N \kappa)^2}}{\kappa^* (\sigma - e_{\tau} e_N e_N \kappa)} = \frac{\sqrt{\tau^2 + (\sigma - e_{\tau} e_N e_N \kappa)^2}}{\kappa (\sigma - e_{\tau} e_N e_N \kappa)}.
\]

Now, we will give an example of quaternionic Bertrand curve of a helix in semi-Euclidean space \( \mathbb{E}^4 \).

**Example 1.** Let \( \alpha \) a unit speed semi-real quaternionic curve in \( \mathbb{E}^4 \)

\[
\alpha(s) = (\cosh s, \sqrt{3} s, \sinh s, s)
\]

for all \( s \in I \). The curve \( \alpha \) is a regular curve and \( s \) is arc-length parameter of \( \alpha \). By considering the equations (4) and (5), we find that Frenet apparatus of \( \alpha \) as follows

\[
T(s) = (\sinh s, \sqrt{3}, \cosh s, 1)
\]

\[
N(s) = (\cosh s, 0, \sinh s, 0)
\]

\[
B(s) = \frac{1}{\sqrt{2}} e_b (2 \sinh s, -\sqrt{3}, 2 \cosh s, 1)
\]

\[
B_4(s) = \frac{1}{\sqrt{2}} e_{\tau} e_n (0, -1, 0, \sqrt{3})
\]

The curvature functions of the unit speed semi-real quaternionic curve \( \alpha \) are \( \kappa(s) = -1 \), \( \tau(s) = \sqrt{2}, (\sigma - e_{\tau} e_N e_N \kappa(s))(s) = 0 \). For \( \lambda = \frac{1}{e_{\tau} e_N} \) and \( \mu = \frac{\sqrt{2}}{e_n} \), the curvatures of \( \alpha \) curve satisfy the relation

\[
\lambda e_{\tau} e_N e_N \kappa(s) + \mu e_n e_N \tau(s) = 1.
\]
So, \( \alpha(s) \) is a Bertrand curve and we can write its Bertrand partner curve \( \alpha^*(s) \) as follows:

\[
\alpha^*(s^*) = (2\cosh s, \sqrt{3}s, 2\sinh s, s)
\]

**Example 2.** We consider a quaternionic curve with arc-length parameter \( s \),

\[
\alpha(s) : I \subset \mathbb{R} \rightarrow \mathbb{E}^4_2 : \alpha(s) = (\cosh s, \sqrt{2}s, \sinh s, \sqrt{2})
\]

for all \( s \in I \), \( \alpha \) is regular curve and its curvature function \( \kappa(s) = -1 \); \( \tau(s) = \sqrt{2} \) and \([\sigma - \epsilon_x e_x \epsilon_N \kappa(s)](s) = 0 \) For \( \lambda = \frac{1}{\epsilon_x \epsilon_N} \) and \( \mu = \frac{\sqrt{2}}{\epsilon_N} \) curvature of \( \alpha(s) \) satisfy the relation

\[
\lambda \epsilon_x \epsilon_N \kappa(s) + \mu \epsilon_n \tau(s) = 1.
\]

So, \( \alpha(s) \) is a quaternionic Bertrand curve and we have its quaternionic Bertrand partner \( \alpha^*(s^*) \) as follows:

\[
\alpha^*(s^*) = (2\cosh s, 2s, \sqrt{2}\sinh s, \sqrt{2}).
\]

**IV. CONCLUSIONS**

\( \alpha \) is a helix (general helix), if its tangent vector \( T \) makes a constant angle with a fixed direction \( U \). If \( \alpha^* \) is a quaternionic Bertrand partner curve of \( \alpha, \alpha^* \) is a helix.

**REFERENCES**
